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An Implicit Finite Difference Method for Eulerian Fluid Dynamics*

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An implicit finite difference scheme is studied for applications to time-dependent flow of a polytropic gas in one dimension. The method is a two-step scheme applied to the Eulerian equations. The scheme is conserving, and can be used for shock smearing applications. A stability analysis demonstrates that the method is unconditionally stable to small disturbances. Computations have been performed with time steps larger than permitted by the CFL condition for a three-point explicit scheme.

1. INTRODUCTION

There are numerous finite difference methods in general use that are suitable for the calculation of time dependent, inviscid, compressible flow fields. In general, such methods smear any shock wave discontinuities over several grid points, and comparison of this dissipative property has been the subject of several detailed surveys [1–3]. The effective shock smearing is of interest since it displays the truncation errors resulting from the difference approximation in an intricate balance with the nonlinearities of the equations of motion [4], and results in a convenient basis for comparison of methods. These surveys show that current difference methods for shock smearing are well developed, useful tools. Since the most widely used methods are explicit, the question arises whether there would be any advantage in developing an implicit scheme. An implicit scheme would couple together all mesh points at the same time level, and therefore, would be expected to relax the necessary stability conditions. This advantage would presumably be blunted by a somewhat larger number of computations required at each mesh point. A second objection is that, should an implicit method use twice the time step of an explicit method, it would have roughly four, eight or sixteen times the time dependent truncation error for first, second or third order accuracy, respectively. Since truncation errors are large at the shock front, fine resolution of the shock could not always be expected for an implicit method operating in a range of step sizes that is econom-

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ical for computation. In the following, an implicit scheme is studied which appears to be economical in a wide range of weak shock applications.

Implicit difference techniques have proven to be successful for quasilinear hyperbolic systems of equations in nonconservation form [5]. In this case, the difference method is developed with the goal of yielding linear difference equations. However, applications of implicit techniques have been more restricted when the equations are of conservation form. For then, the difference equations yield a simultaneous system of nonlinear equations coupling the values at all grid points at the new time level. The basic approach is to solve this system of equations by an iterative technique. In terms of the difference method applied to the differential equations, this iteration can be viewed as a predictor–corrector method. The earliest and most widely studied approach was initiated by Gary [6, 7] in which an explicit predictor was followed by a stabilizing, implicit corrector which was iterated upon a specified number of times. The predictor was the unconditionally unstable scheme formed by centered space differences and forward time differences, and the overall scheme was stable or unstable depending upon the number of iterations that were made on the corrector at each grid point. The explicit predictor approach was refined by Abarbanel and Zwas [8] to include the use of the Lax–Wendroff predictor with artificial viscosity. Recently Gourlay and Morris [9] proposed an implicit corrector which does not require iteration and which maintains conservation form.

As an alternative to the above methods, an implicit predictor can be used in conjunction with the corrector. The corrector, then, is not necessary for stability, and can be used to increase the order of accuracy as was suggested by Douglas [10]. Or, the corrector can be used to restore conservation as was suggested by Yanenko and Yaushev [11] for the Lagrangian form of the equations of motion. The central idea of the method is to obtain an implicit predictor by quasilinearization of the differential equations. The resulting system of difference equations is linear and unconditionally stable. This method has the advantage of not requiring the corrector for stability, and hence the corrector need only be applied once. The power of the method lies in the freedom to choose any implicit predictor that has been developed for quasilinear hyperbolic equations.

Two variations of a scheme for implementing the implicit predictor–corrector technique are presented below. Both methods are applied to the Eulerian equations describing one dimensional, time dependent flow of a polytropic gas of exponent γ. The implicit predictor is based on the normal or characteristic form of the equations and employs one-sided differences along the lines first suggested by Courant, Isaacson and Rees [12] for explicit methods. Overall conservation is restored by the second or leap-frog step. The resulting difference equations are first or second order accurate depending upon the size of the time step used in the predictor relative to the time step used in the corrector. This ratio will be denoted by θ. The effect of various time steps for the predictor [9] and of various averaging methods [1] of the

predictor values with the current time level data have been extensively studied for the two step version of the Lax–Wendroff method [13]. Such studies have shown that the shock smearing behavior may be widely modified and improved by the choice of the averaging technique. Similar freedom is available in the present method, but the great number of possible choices has not been investigated.

In addition, it should be noted that the extension of the one dimensional scheme to multidimensional problems can be formally achieved by fractional step techniques. Splitting formulas of first, [14] second [15] and third [16] order accuracy are available. As noted by Burstein and Mirin [16], it would be advantageous to use an implicit operator for one direction in a multidimensional splitting scheme when that direction is primarily responsible for stability restrictions. Hence one dimensional implicit schemes of the type studied here may lead to mixed implicit–explicit schemes with greatly relaxed stability requirements for multidimensional problems. Use of such schemes has not yet been attempted to the author's knowledge.

2. Difference Method

The predictor is based on the characteristic form [17] of the equations of motion, a selection also made by Yanenko and Yaushev [11]. The characteristic form is computationally more convenient than the quasilinear convective form of the Euler equations. Since the equations of the system are uncoupled in the derivatives, the coefficient matrix corresponding to each family of the characteristic system may be inverted separately without resorting to block matrix techniques. In addition, the characteristic form permits easy inclusion of the boundary conditions. Since the equations of motion are redundant at the boundary to admit the boundary conditions, only certain linear combinations of the momentum and continuity equations remain of use there. These linear combinations are equations of the characteristic system. Hence at the boundary there is a one to one correspondence with the boundary conditions, and the resulting system of equations is solved simultaneously.

In conservation form, the equations of motion are given by

$$w_t + f_x = 0, \tag{1}$$

where w and f are vector-valued functions

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{pmatrix}, \tag{2), (3}$$

with $E = e + \frac{1}{2}u^2$, e is the internal energy of the gas and $H = E + p/\rho$. The

characteristic form of these equations is derived by diagonalizing $A(w)$, the Jacobian matrix of f with respect to w. These equations have the simple form for a gas with polytropic exponent γ,

$$v_t + A'v_x = 0, \tag{4}$$

where

$$v = \begin{pmatrix} r \\ s \\ S \end{pmatrix}, \qquad A' = \begin{pmatrix} u + c & 0 & -c^2/\gamma \\ 0 & u - c & c^2/\gamma \\ 0 & 0 & u \end{pmatrix}, \tag{5), (6}$$

where $r = [2/(\gamma - 1)]\, c + u$ and $s = [2/(\gamma - 1)]\, c - u$ are the Riemann invariants, and S, the entropy function. The nondiagonal appearance of (5) and (6) is due to the introduction of the Riemann invariants when the flow is nonisentropic.

Let h and τ denote the increments in the x and t coordinates respectively, and let subscripts indicate the value at the grid point and superscripts indicate the value at the time level. At the nth time level, the value of w at the kth grid point is w_k^n. The second step of the method uses the leap-frog difference expression applied to Eqs. (1)–(3),

$$w_k^{n+1} = w_k^n - (\tau/h)[f(w_{k+1/2}^{n+\theta}) - f(w_{k-1/2}^{n+\theta})]. \tag{7}$$

The quantities $w_{k+1/2}^{n+\theta}$ are subsidiary values determined by the first step, where θ is a number between 0 and 1 denoting the fraction of the total time step τ that was used during the predictor step. Hence $\theta\tau$ is the predictor time step, and $\theta = \frac{1}{2}$ represents second order accuracy of the overall method.

The implicit step is used to advance the variables at the time layer t^n to their values at the time layer $t^{n+\theta} = t^n + \theta\tau$. The scheme uses one sided differences applied to the characteristic form of the equations. Let Δ_+ and Δ_- denote forward and backward differences respectively, and let Δ denote the following general upwind difference operator

$$\Delta = \tfrac{1}{2}[(\Delta_+ + \Delta_-) - \operatorname{sign}(A_m')(\Delta_+ - \Delta_-)], \tag{8}$$

where sign (A_m') has the value of $+1$ or -1 respectively when the particular diagonal element of A' is positive or negative. Since

$$\Delta_+ - \Delta_- = \Delta_+\Delta_- = \Delta_-\Delta_+$$

the difference operator (8) always introduces a truncation error which is a second difference of indefinite sign. In the full set of equations, the coefficient of the second difference is the positive quantity $A_m' \operatorname{sign}(A_m')$, and hence the scheme is dissipative.

The difference equations are therefore

$$v_{k+1/2}^{n+\theta} = \hat{v}_{k+1/2}^{n} - \theta(\tau/h)\, A_{k+1/2}'^{n}\, \Delta v_{k+1/2}^{n+\theta}\,. \tag{9}$$

The quantity $\hat{v}_{k+1/2}^{n}$ is not directly available from the data provided by the corrector. The two schemes that are studied differ in the method of evaluation of this quantity. For scheme I, $\hat{v}_{k+1/2}^{n}$ is assigned the value of either $v_k{}^n$ or v_{k+1}^n depending upon the value of sign(A_m'). This switching operator can be written in terms of averaging and central difference operators. Let μ denote the averaging operation

$$\mu v_{k+1/2} = \tfrac{1}{2}(v_{k+1} + v_k),$$

and let Δ_0 denote the central difference operation

$$\Delta_0 v_{k+1/2} = v_{k+1} - v_k\,.$$

Then the operator denoting the selection process will be given by

$$\Lambda = \mu - \tfrac{1}{2}\,\mathrm{sign}(A_m')\,\Delta_0\,. \tag{10}$$

The operator Λ, if used without the implicit term of Eq. (9), represents the explicit finite difference method proposed by Godunov [13]. Therefore, the predictor

$$v_{k+1/2}^{n+\theta} = \Lambda v_{k+1/2}^{n}$$

represents a conditionally stable method when used in conjunction with Eq. (7).

Use of the Scheme I expression

$$\hat{v}_{k+1/2}^{n} = \Lambda v_{k+1/2}^{n} \qquad \text{(SCHEME I)} \tag{11}$$

in Eq. (9) does not yield a predictor of first order accuracy. To achieve first order accuracy, the term $\frac{1}{2}\,\mathrm{sign}(A_m')\,\Delta v_{k+1/2}^{n+\theta}$ must be added to Eq. (9) to cancel the contribution from the second term of Eq. (10). This modification is only required in Scheme I.

The evaluation of $\hat{v}_{k+1/2}^{n}$ used in Scheme II is much more direct, being given by the average of the adjacent mesh points,

$$\hat{v}_{k+1/2}^{n} = \mu v_{k+1/2}^{n}\,. \qquad \text{(SCHEME II)} \tag{12}$$

The predictor equation resulting from the substitution of (12) into Eq. (9) is first-order accurate.

The remaining quantities that are not directly defined by the data at the mesh points are the non zero elements of each matrix $A_{k+1/2}'^{n}$. For both schemes,

this evaluation was made in the manner consistent with the previous evaluations of $\hat{v}^n_{k+1/2}$, that is

$$A'^n_{k+1/2} = A'(\hat{v}^n_{k+1/2}), \qquad \text{(SCHEMES I, II)} \tag{13}$$

where A' is the matrix function defined by Eq. (6), and $\hat{v}^n_{k+1/2}$ is determined by the corresponding Eq. (11) or (12). Since Λ in equation (11) depends upon sign(A_m'), this last evaluation is formally implicit for Scheme I. However no difficulty was encountered in making the equation explicit by assigning values to sign(A_m') as indicated by the neighboring mesh points.

3. Solution of Difference Equations

In the computed examples, the wall or closed end is taken at the left-hand end of the mesh and the right-hand end of the mesh is assumed to be an open-ended tube. Let the wall be situated between $k = 0$ and $k = 1$, then the condition $u = 0$ requires that

$$r^{n+\theta}_{1/2} - s^{n+\theta}_{1/2} = 0.$$

This equation replaces the equation of the r-family, since the domain of dependence of the r-equation (the first component of the vector equation (9) at the wall) lies beyond the end of the grid; however, the s-equation (the second component of vector equation (9) at the wall) remains valid

$$s^{n+\theta}_{1/2} + \frac{\xi'}{1-\xi'} s^{n+\theta}_{3/2} = \frac{1}{1-\xi'} [s^n_{1/2} - c^2/\gamma(S^{n+\theta}_{3/2} - S^{n+\theta}_{1/2})],$$

where ξ' is the coefficient of the difference operator in the second component of Eq. (9).

The right hand boundary will be situated between $k = K$ and $k = K + 1$. If the inflow is subsonic, the r-equation can be applied at the end of the grid

$$r^{n+\theta}_{K+1/2} - \frac{1}{1+\xi} r^{n+\theta}_{K-1/2} = \frac{1}{1+\xi} [r^n_{K+1/2} + c^2/\gamma(S^{n+\theta}_{K+1/2} - S^{n+\theta}_{K-1/2})],$$

where ξ is the coefficient of the difference operator in the first component of Eq. (9). The second condition is taken to be the constant pressure condition,

$$r^{n+\theta}_{K+1/2} + s^{n+\theta}_{K+1/2} = [4/(\gamma - 1)]\, c_\infty ,$$

applicable for an open ended tube, and which replaces the s-equation at this point.

To avoid the necessity of employing block matrix techniques the r- and s-systems

can be viewed as coupled only by a single parameter, while the S system is entirely uncoupled. Dropping $\frac{1}{2}$ from the subscripts and dropping superscripts altogether, the r-system can be written:

$$\begin{aligned} r_0 - s_0 &= 0, \\ -a_k r_{k+1} + b_k r_k - c_k r_{k-1} &= d_k, \qquad 1 \leqslant k \leqslant K-1, \\ b_K r_K - c_K r_{K-1} &= d_K, \end{aligned}$$

which is a tridiagonal system of $K+1$ equations for the $K+1$ unknowns r_k, with a parametric dependence on s_0. Similarly the s-system is

$$\begin{aligned} -a_0' s_1 + b_0' s_0 &= d_0', \\ -a_k' s_{k+1} + b_k' s_k - c_k' s_{k-1} &= d_k', \qquad 1 \leqslant k \leqslant K-1, \\ r_K + s_K &= d_K', \end{aligned}$$

which is a tridiagonal system of $K+1$ equations for the $K+1$ unknowns s_k, with a parametric dependence on r_K.

These coupled tridiagonal systems are solved by an extension of the method of gaussian elimination recommended by Richtmyer [13]. The solution to the r-system will be found from the recursion formula

$$r_k = e_k r_{k+1} + f_k + g_k s_0, \tag{14}$$

which represents a backward substitution process. The parameters e_k, f_k, and g_k are found recursively by a forward elimination process starting from the first equation and finishing with the Kth equation. The recursive evaluation of r_k from (14) by back substitution cannot be started immediately after all the e, f and g are found since s_0 is still unknown. Combining the substitution equation (14) and the Kth equation of the system, gives one of the expressions

$$r_K = f_K + g_K s_0, \tag{15}$$

which couple the r and s systems together.

For the s-system the elimination and substitution are performed in the opposite direction to the r-system. Thus a forward substitution formula is assumed:

$$s_k = e_k' s_{k-1} + f_k' + g_k' r_K. \tag{16}$$

The values of e_k', f_k' and g_k' are found recursively by a backward elimination process starting with the Kth equation and finishing with the first equation. Combining the substitution equation (16) and the first equation of the s-system results in the second coupling equation to be satisfied simultaneously with Eq. (15)

$$s_0 = f_0' + g_0' r_K. \tag{17}$$

Hence after elimination has been completed on both systems of equations, the simultaneous set (15) and (17) are solved to determine s_0 and r_K. Once s_0 and r_K are known, the substitution formulas (14) and (16) can be evaluated recursively for the remaining unknowns r_k and s_k, respectively.

There are only four additional multiplications and two additional divisions required at each grid point over that required by an uncoupled system. This procedure is much more efficient than would be required if the predictor were not in characteristic form. In that case a nine-diagonal wide band matrix would have to be inverted.

4. Stability Analysis

In many applications of finite difference methods to hydrodynamic problems, stringent stability conditions may have to be satisfied in some retricted part of the flow field. This situation may arise, for example, when a mesh of nonuniform spacing is employed, or when the Jacobian of the transformation to a mesh of uniform spacing becomes small. As a consequence, the time step will have to be much smaller than the desired interval of data output. For multidimensional problems based on splitting techniques, it has been demonstrated [18] that the time step should be of nearly the same step size as the data output interval. A reduction in the severity of the stability condition would be of importance to applications of this type.

The principal limitation on the time step size is the CFL requirement that the domain of dependence of the difference scheme contain the domain of dependence of the partial differential equations. The relaxation of the stability requirement for an implicit scheme is due to the extension of its domain of dependence to the entire flow region, while an explicit three-point scheme can only have a domain of dependence of $2h$. In practice the larger domain of dependence only achieves a limited extension for the highly nonlinear problem. The procedure used in linearizing the difference equations must fail with large enough time steps, and it is difficult to say for exactly what time step it will fail.

The linear stability analysis is based on the Fourier technique for the system of linearized equations. Linearizing about the uniform state $\bar{u}$, $\bar{p}$, $\bar{\rho}$ yields the simplified system of perturbation equations

$$\begin{aligned} \rho_t' + \bar{u}\rho_x' + \bar{\rho}u_x' &= 0, \\ u_t' + \bar{u}u_x' + (1/\bar{\rho})\,p_x' &= 0, \\ e_t' + \bar{u}e_x' + (\bar{p}/\bar{\rho})\,u_x' &= 0, \end{aligned}$$

where primes denote a perturbation from the uniform state. Substituting that

$e' = e_p p' + e_\rho \rho'$, and forming certain linear combinations of these perturbation equations yields a differential system of the form

$$v_t + \bar{A} v_x = 0,$$

where

$$v = \begin{pmatrix} p'/\bar{\rho}\bar{c} + u' \\ p'/\bar{\rho}\bar{c} - u' \\ p' - \bar{c}^2 \rho' \end{pmatrix}, \qquad A = \begin{pmatrix} \bar{u} + \bar{c} & 0 & 0 \\ 0 & \bar{u} - \bar{c} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}.$$

The corresponding difference equations are

$$v_k^{n+1} = v_k{}^n - (\tau/h)\, \bar{A}(v_{k+1/2}^{n+\theta} - v_{k-1/2}^{n+\theta}). \tag{18}$$

Each equation of this system corresponds to precisely one equation of the predictor system and hence the stability question can be investigated very simply. Let β denote the value of the coefficient of the difference term in Eq. (18). For $\beta > 0$ the corresponding predictor for Scheme I is given by

$$v_{k+1/2}^{n+\theta} - \frac{\xi}{1+\xi} v_{k-1/2}^{n+\theta} = \frac{1}{1+\xi} v_k{}^n \tag{19}$$

where ξ denotes the coefficient of the difference term in (9). When $\beta < 0$, the predictor is

$$v_{k+1/2}^{n+\theta} + \frac{\xi}{1-\xi} v_{k+3/2}^{n+\theta} = \frac{1}{1-\xi} v_{k+1}^{n}. \tag{20}$$

Scheme II is treated by replacing $v_k{}^n$ and v_{k+1}^n by $\frac{1}{2}(v_k{}^n + v_{k+1}^n)$ in Eqs. (19) and (20) respectively.

To determine the stability limits for Scheme I equations, the Fourier representation is assumed, where λ is the complex amplification factor and k is the wave number of the harmonic disturbance

$$v_k{}^n = \lambda^n e^{ik\varphi}.$$

If the harmonic disturbance is to be damped, then $|\lambda| < 1$ is a necessary condition. Assume that β is positive, then Eq. (19) requires that the intermediate values from the implicit step are given by

$$v_{k+1/2}^{n+\theta} = \frac{1}{1+\xi-\xi e^{-i\varphi}} \lambda^n e^{ik\varphi}.$$

Substitution of this result into Eq. (18) will show that the amplification factor λ for the total scheme satisfies the relation

$$\lambda = 1 - \beta \frac{1 - e^{-i\varphi}}{1 + \xi(1 - e^{-i\varphi})} = \frac{1 + (\xi - \beta)p}{1 + \xi p},$$

where $p = 1 - e^{-i\varphi}$. It is easy to show that the amplification factor magnitude is given by the result

$$1 - |\lambda|^2 = \frac{2\beta(1 + 2\xi - \beta)(1 - \cos\varphi)}{1 + 2\xi(1 + \xi)(1 - \cos\varphi)}.$$

Since the trigonometric polynomial is positive, the scheme is stable for all time step values provided that $1 + 2\xi - \beta > 0$, or from (9) when

$$\theta \geqslant \tfrac{1}{2} \qquad \text{when} \quad \beta > 0,$$

when $\theta < \frac{1}{2}$ the amplification factor is greater than unity and disturbances are magnified. A similar analysis applied to Eq. (19) gives

$$\theta \geqslant \tfrac{1}{2} \qquad \text{when} \quad \beta < 0.$$

The analysis is only trivially different for Scheme II, where it is found again that $\theta \geqslant \frac{1}{2}$ is necessary for stability. The value of θ for second order accuracy is seen to be at the stability boundary. An excess of damping is achieved by using an implicit step that is larger than the value for best accuracy.

5. Results

Calculations of the propagation of a moderately strong shock wave, pressure ratio 5, in a gas of polytropic exponent, $\gamma = 1.4$, were performed with various time steps. The overall time step is τ and is indicated in the figures through the Courant number,

$$\text{Courant number} = (|u| + c)_{\max} \frac{\tau}{h},$$

which has the value of at most unity for an explicit three point difference method. The ratio of the predictor to the corrector time step is the quantity θ.

Figures 1–5 demonstrate the behavior of Scheme 1 for increasing values of the Courant number. When the Courant number was close to one, the predictor–corrector time step ratio θ could be reduced to its minimum value of 0.5. Figures 1 and 2 show these results for Courant numbers of 1.0 and 1.3. From the monotonic results in Figure 1, it is apparent that the scheme was operating with first order

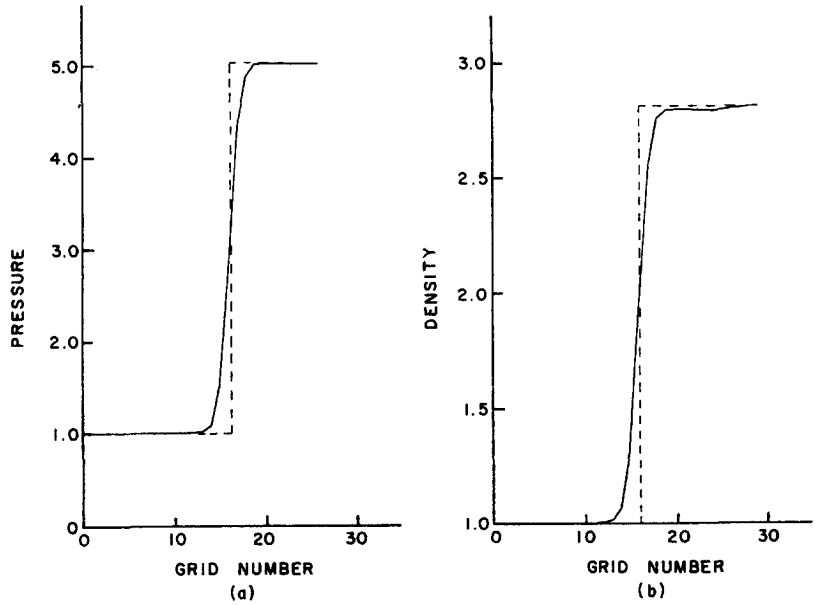

FIG. 1. Shock profile, for shock pressure ratio 5.0, from computation with Courant number = 1.0, theta = 0.5 and Scheme I: (a) pressure, (b) density. Numerical results ———, theoretical values – – –.

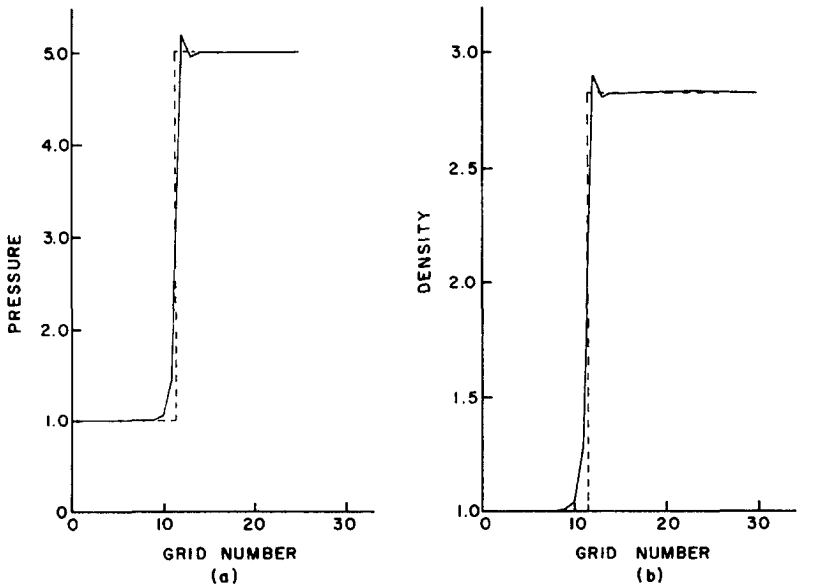

FIG. 2. Shock profile, for shock pressure ratio = 5.0, from computation with Courant number = 1.3, theta = 0.5 and Scheme I: (a) pressure, (b) density. Numerical results ———, theoretical values – – –.

accuracy, and this is attributed to the use of the requirement that $\theta\tau/h \mid A_m' \mid$ be greater than 0.5. These conditions require that the scheme becomes Godunov's method for small enough time steps. At a Courant number of 1.5 or greater, which is a 50% larger time step than permitted by explicit techniques, Scheme I would operate in a stable manner only with values of $\theta > 0.5$.

In Fig. 3, the calculation of reflection of an incident shock with pressure ratio 5.0

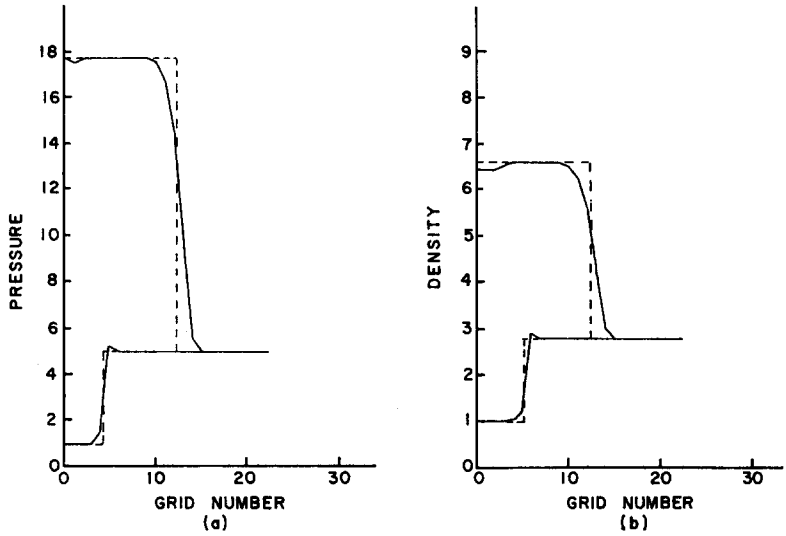

FIG. 3. Shock reflection for an incident shock wave of pressure ratio = 5.0. Computation by Scheme I with Courant number = 1.3, theta = 0.5: (a) pressure, (b) density. Numerical results ———, theoretical values – – –. Theoretical reflected shock pressure is 17.7.

from a wall is shown. The wall is located at the left end of the grid with the wave incident from the right. Behind the reflected shock, the pressure at the wall agrees with theory [19] to within 2%. The shock speed is also within 2% of theory. The predictor–corrector time step ratio, θ, was 0.5, and the overall time step corresponded to a Courant number of 1.3 as in Fig. 2. This same reflection is shown as computed with a Courant number equal to 2.0 in Fig. 4. Weaker shock reflection problems have been successfully calculated at much larger time steps. Computations for weak shock problems are more successful since the characteristic form used for the predictor is more accurate for weak waves. For example, in Fig. 5 results are shown for reflection of a shock which has an incident pressure ratio of 2.0. This calculation was made with a Courant number of 3.0.

Figure 6 shows results from calculations made with Scheme II. Pressure profiles

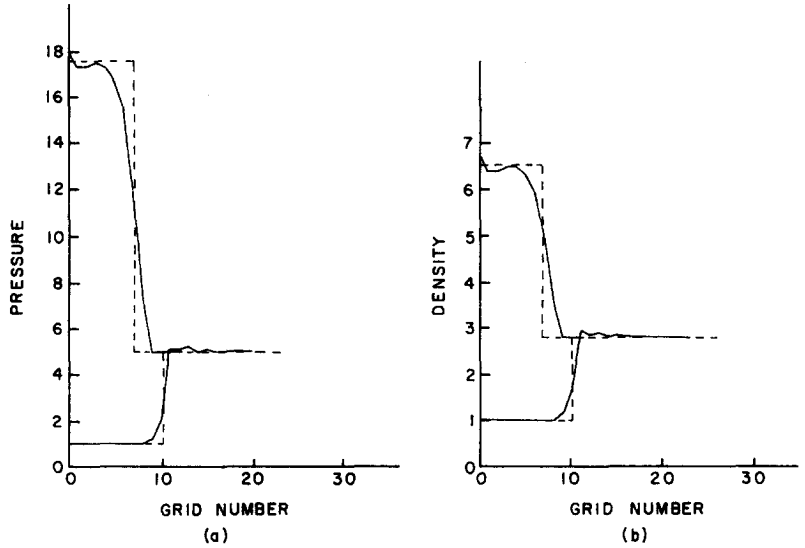

FIG. 4. Shock reflection for an incident shock wave of pressure ratio 5.0. Computation by Scheme I with Courant number = 2.0, theta = 1.0: (a) pressure, (b) density. Numerical results ———, theoretical values – – –. Theoretical reflected shock pressure is 17.7.

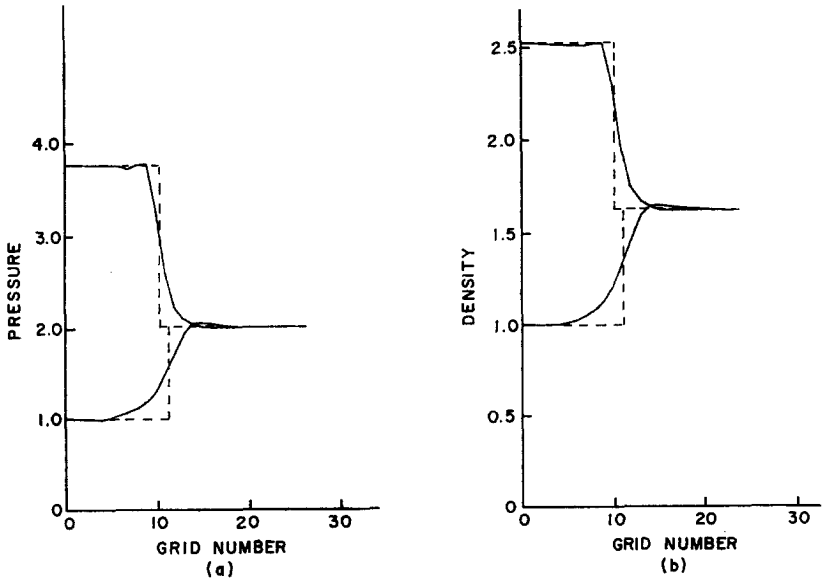

FIG. 5. Shock reflection for an incident shock wave of pressure ratio = 2.0. Computation by Scheme I with Courant number = 3.0, theta = 1.0: (a) pressure, (b) density. Numerical results ———, theoretical values – – –. Theoretical reflected shock pressure is 3.75.

are shown for two problems, shock propagation and shock reflection. For the pressure ratio 5 shock propagation, a comparison with Fig. 4 graphically demonstrates the differences between Scheme I and Scheme II. Scheme II always displays a much smoother and more smeared profile than does Scheme I. Although Scheme II appears smoother and more dissipative than Scheme I, it becomes unstable under a wider range of θ and τ than Scheme I. These highly smeared profiles are also evident in the weak shock reflection seen in Fig. 6.

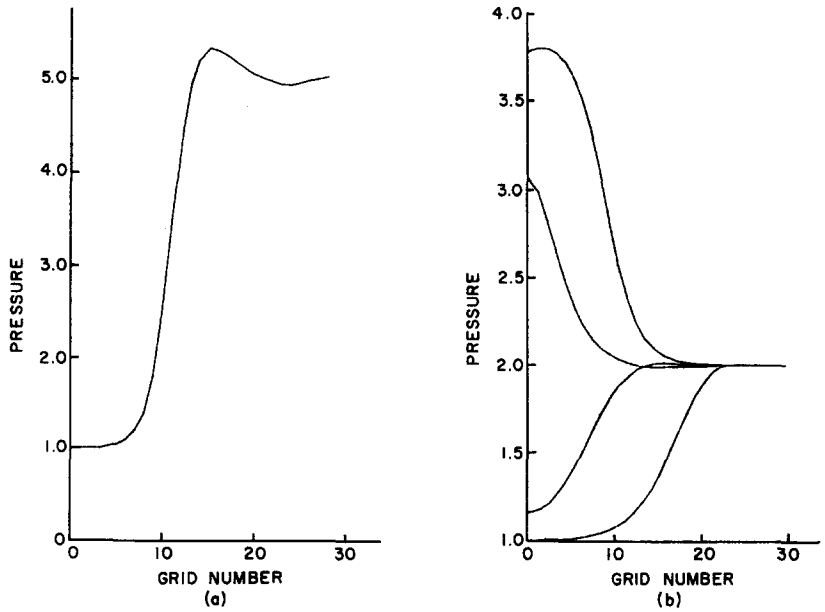

FIG. 6. Computed shock profiles obtained for Courant number = 2.0, theta = 1.0, Scheme II: (a) shock propagation pressure profile for shock of pressure ratio = 5.0 (b) shock reflection pressure profiles for an incident shock of pressure ratio = 2.0.

Of the two schemes, Scheme I appears to be superior on the basis of the tests. This scheme has been used for strong shock reflection studies and is found to operate in a stable manner, although the profiles can become ragged. On the other hand, when time steps are used that are comparable to the largest permissable with explicit methods, the scheme produces very smooth clean results. Perhaps the best performance of the method is in the transonic flow regime, where weak shock waves have been computed smoothly and cleanly at very large time steps.

The computational labor involved in inverting the tridiagonal matrices at each time step is somewhat larger but comparable to that involved in evaluation of a sophisticated artificial viscosity. The method appears to run about 25%

slower than the Richtmyer version of the Lax–Wendroff method with artificial viscosity. This comparison must be qualified as resulting from two widely differing programs. But on this basis the method would be economically attractive for Courant numbers on the order of 1.3 or greater.

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